

A Neuroeconomic Theory of Memory Retrieval

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Abstract

We propose a theory of “optimal memory management” that unveils causal relationships between memory systems and the characteristics of the information retrieved. Our model shows that if the declarative memory is more accurate but also more costly than the procedural memory, then it is optimal to retrieve exceptional experiences with the former and average experiences with the latter. The theory offers some other testable predictions: (i) decisions are closer to original experiences when the declarative memory is invoked; (ii) a memory system is more likely to be employed to encode and retrieve information if its accuracy increases or its cost decreases; and (iii) the declarative system is more likely to be invoked when the importance of recalling information accurately increases.

Keywords: memory systems, memory management, declarative, procedural, neuroeconomic theory.

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1 Introduction

Bounded memory is arguably one of the most important limitations in humans, an aspect that has received considerable attention from researchers. Formal mathematical models of limited memory can be found in fields as diverse as statistics (see e.g., Cover and Hellman (1970)), artificial intelligence (see e.g., Narendra and Thathachar (1989)), psychology (see e.g., Anderson and Milson (1989)), computation theory (see e.g., Feder (1991)) and economics (see e.g., Kocer (2011)), to name a few. However, in order to concentrate on other aspects, they all abstract from an important result in neuroscience: memories can be encoded by different systems and each system has some special properties. The goal of this paper is to build (to our knowledge) the first mathematical model of bounded memory where experiences are optimally encoded by different systems depending on their characteristics. More precisely, our model unveils *causal* relationships between the memory system employed and the type of the information retrieved. To better understand the main building blocks of the theory, we first start with a brief overview of the existing neurophysiological evidence on memory.

Memory refers both to the conscious recollection of facts and historic events and also to the unconscious and automatic retrieval of information necessary to perform some habitual actions. However, the processes involved in storing, learning and retrieving these different types of information differ largely. The literature in neuroscience reports findings indicating the existence of different memory systems in the brain (see e.g. Packard et al. (1989), and Poldrack and Foerbe (2008) for a review). This coexistence of multiple memory systems can be supported on evolutionary grounds (Sherry and Schacter (1987)). An accurate classification of memory systems has been obtained by correlating the types of information memorized with the underlying biological mechanisms involved in the memory processes (see e.g. Squire (2004) for a review). Memory can be broadly classified into two main classes.

Declarative memory refers to the capacity to recollect information in a conscious way. It is based on the ability to detect and encode what is unique about an event (Ullman (2004)). Learning occurs fast (with few exposures) and the learnt material is consciously known and easily verbalized. Learning is effortful and engages working memory resources (Craik et al. (1996)). Knowledge acquired with the declarative memory system is flexible and can be used in a variety of contexts, but it also tends to erode. Declarative memory engages the hippocampus and surrounding structures (Eichenbaum (2001)). These structures are involved in the formation of memories but also in the ability to retain and recall them (Gabrieli and Kao (2007)). The lateral Prefrontal cortex (PFC) is engaged in the memory process of contextual details of an experience. The Left Dorsolateral PFC is activated when memories are formed while the Right Dorsolateral PFC is activated when memories are retrieved (Kapur et al. (1997)). The amygdala is involved in the encoding

and retrieval of emotionally charged memories (Adolphs et al. (1997), Sharot et al. (2007)).

Non-declarative memory refers to learnt skills and habits but also to perceptual learning or conditioning. Non-declarative memory detects what is common to several situations. Learning is gradual and slow, the decision-maker learns through trial-and-error, and requires feedback. The learnt material is also unconscious and difficult to verbalize. Learning requires effortless attention. Learned knowledge is rigid, used in specific contexts, and durable. It engages a variety of structures depending on the finer subclassification of memories. For example, skills and habits are placed under the umbrella of procedural memory and engage structures like the striatum (Kreitzer (2009)). Also, conditioning is linked to the amygdala and the cerebellum (see Squire (2004) for a detailed classification).

This classification suggests a connection between the systems involved and the type of information memorized. We can think of the different memory systems as tools to solve different problems. For instance, the declarative system helps find a solution to problems like “where did I park today?” while the procedural system solves best problems like “where do I usually park?”. The relationship between memory systems and types of memories is still under study. Yet, studies in the last few years provide interesting results.

Firstly, memory systems are *substitutable*. Bayley et al. (2005) show that subjects with impaired procedural memory improve over time their performance in the weather prediction task by repeatedly exercising their declarative memory, even this is a paradigmatic example where procedural memory is most adequate. Also, what is learnt depends crucially on which system is engaged (Bayley and Squire (2002)). In particular, when one system is impaired, a different system may ‘replace it’ and be used to process memories that otherwise engage the impaired system (Dagher et al., 2001). However, using a system not designed for the task has detrimental effects on performance. For instance, a subject suffering from amnesia may often fail to remember a past fact because the system engaged is non declarative. Overall, each system is tailored to certain memories. In the economic jargon, systems are only ‘imperfect substitutes’ (see e.g. Poldrack and Packard (2003)).

Secondly, systems are selected depending on task demands. In particular, there is evidence that neurobiological mechanisms are in place to make sure behavior is *optimized*, that is, to employ the memory system most suitable to the experience (Poldrack et al. (2001), Poldrack and Rodriguez (2004), Foerde et al. (2006)).

Substitutability and optimization are key properties in a decision making situation. The evidence reviewed here suggests that the resort to a given memory system is an *endogenous decision*: (i) several systems can be employed to retrieve memories, (ii) different systems have different properties which make them suitable for the encode and retrieval of different experiences, and (iii) the choice of one system over another will be the result of an optimization process. Starting from these premises, the purpose of this study is to

build a theory of optimal *memory management*, that is, one that predicts the endogenous choice between competing memory systems as a function of the experience to memorize.

To facilitate the exposition, we consider a simple and abstract thought experiment, in which a decision-maker (hereafter DM) learns a piece of information relevant for future choices. Unfortunately, DM has imperfect memory so the exact information received may not be correctly recalled at the time of the decision. For simplicity, we assume that the information is stored and can be retrieved using either the *declarative* memory system or the *procedural* memory system. We investigate the optimal information retrieval strategy as a function of the environment.

Following the evidence, we work under the hypothesis that systems differ in their *accuracy* and *cost*. Accuracy corresponds to the degree to which DM can recover the precise experience, while cost refers to the working memory resources needed for the retrieval of information. In line with the evidence reviewed above, we assume that the declarative memory system produces more accurate representations of the experience but is also more costly than the procedural memory system. Although these systems differ in many other respects as well and some other systems are also at play in the retrieval of memories, our theory focuses exclusively on those two characteristics in order to better assess their impact on behavior. The reader shall therefore keep in mind that other important effects, emerging from other features, cannot be predicted by our model.¹

2 Model

To formalize our thought experiment, we consider a four-stage decision-making problem.

In stage 1, DM acquires information about the state of the world. Let $x \in \mathbb{R}$ be the state learned by the individual, and denote by X the real-valued random variable from which x is drawn. We assume that X follows a normal distribution. Formally:

$$X \sim \mathcal{N}\left(\mu, \frac{1}{p}\right),$$

where μ is the mean of X , and p is the inverse of the variance of X , also called “precision”.

In stage 2, a memory about the state is formed. DM can invoke the declarative memory system ($i = D$) or the procedural memory system ($i = P$). This choice impacts future memory recollections.

¹We realize that by neglecting many important characteristics of these and other memory systems, our theory can be seen as simplistic and reductionist. This modeling strategy, standard in economic theory, is chosen on purpose. Using such a simple model allows to draw sharp conclusions. The effects highlighted will still be valid in a more general framework unless the premises of our model (different accuracy and cost of declarative and procedural memories) are either inaccurate or of second-order importance.

In stage 3, the state is noisily recollected. If memory system i ($\in \{D, P\}$) was invoked, the individual retrieves the following signal s_i correlated with the true state x :

$$s_i = x + u_i \quad \text{where} \quad u_i \sim \mathcal{N}\left(0, \frac{1}{h_i}\right).$$

The added noise u_i follows a normal distribution with precision h_i . In expectation, the signal is accurate $E(s_i) = x$. In order to capture the greater accuracy of information retrieval under the declarative system than under the procedural system, we assume that $h_D > h_P$ (notice that $h_i = +\infty$ implies perfect recollection of the state whereas $h_i = 0$ implies no recollection whatsoever). Also, invoking memory system i has a cost c_i . Following the evidence previously described, we assume that the declarative system involves a higher cost than the procedural system, namely $c_D > c_P$.

In stage 4, DM undertakes an action a and the payoff he obtains from the decision depends on the congruence between the action and the state. For simplicity, we assume that the payoff $l(a, x)$ is given by a standard quadratic utility loss:²

$$l(a, x) = -\beta(a - x)^2$$

with $\beta > 0$. According to this formulation, if the state x is recalled with exactitude, the individual's optimal action is:

$$\tilde{a}(x) = \arg \max_a l(a, x) \quad \Rightarrow \quad \tilde{a}(x) = x.$$

Deviations from $\tilde{a}(x)$ imply a loss which is increasing in β . The parameter β thus represents the importance of the decision or the sensitivity of DM to losses.

A simple example to illustrate this sequence of events is that of a DM recalling how much he liked a product before purchasing it again. His experience in stage 1 (the state) reveals the optimal quantity x he should purchase. In stage 3, he forms a recollection of the experience (the signal, s_i) but it will be distorted due to imperfect memory (the noise, u_i). Based on such recollection, DM in stage 4 may decide to purchase (the action, a) too much or too little of the product, that is, $a \gtrless x$. Both deviations imply a utility loss.

Notice that the declarative system is tailored to answer the question “how much do I like this particular product?”. If invoked, it is likely that DM will recall a signal s_D close to the experience x and answer the question correctly. However, precision comes at a cost, and working memory is highly involved in the process. On the other hand, the procedural system is designed to answer the more general question “how much do I like this type of product in general?”. If it is engaged, the memory recollection s_P is likely to

²The reader should not worry that the final utility is negative: adding a positive constant term to the utility would not affect the analysis.

be farther away from the experience x , as DM will miss specific features of the product and focus instead on general characteristics he usually likes (and that may or may not be the distinctive feature of that product). The benefit, however, is that this memory retrieval requires little effort.

The decision process is summarized by the following timeline.

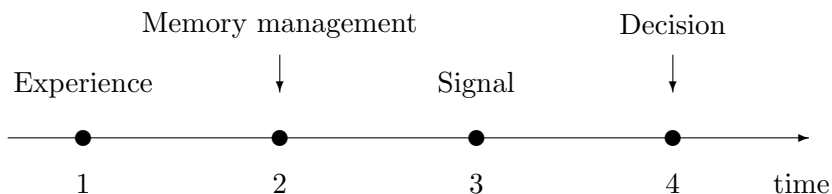


Figure 1. Timing

To solve this decision problem, we proceed by backward induction as usual in microeconomic theory. If memory system i is chosen in stage 2 and signal s_i is retrieved in stage 3, then DM in stage 4 should choose the action that maximizes his expected payoff:

$$\hat{a}(s_i) = \arg \max_a - \int_x \beta (a - x)^2 dF_i(x | s_i)$$

where $F_i(x | s_i)$ is the revised distribution of the state x given the memory system i , the signal retrieved s_i and the prior distribution of states X . Since the objective function is concave, the optimal action satisfies the first-order condition, namely:

$$\hat{a}(s_i) = E_i[X | s_i],$$

where E_i is the expectation operator. In words, DM's optimal action is simply his expected belief about the state given the signal retrieved.

In stage 2, that is after observing the state x , DM knows that if system i is invoked, he will obtain in stage 3 a signal s_i drawn from $G_i(s_i | x)$. Given that signal, he will undertake in stage 4 the action $E_i[X | s_i]$. Therefore, DM's expected payoff in stage 2 is:

$$V_i(x) = - \int_{s_i} \beta (E_i[X | s_i] - x)^2 dG_i(s_i | x) - c_i$$

The normality assumptions of state and signal imply that:

$$s_i | x \sim \mathcal{N}\left(x, \frac{1}{h_i}\right) \quad \text{and} \quad X | s_i \sim \mathcal{N}\left(\frac{p}{p+h_i} \mu + \frac{h_i}{p+h_i} s_i, \frac{1}{p+h_i}\right)$$

Substituting in the previous equation, we then get:

$$V_i(x) = -\frac{\beta h_i}{(p+h_i)^2} - \frac{\beta p^2}{(p+h_i)^2} (x - \mu)^2 - c_i, \quad i \in \{D, P\} \tag{1}$$

Finally, in stage 2 it is optimal to employ the memory system i that achieves the highest expected payoff, namely:

$$i^* = \arg \max_{i \in \{D, P\}} V_i(x)$$

3 Results

We can now review the properties of DM's decision in stage 4 given the memory system invoked, as well as DM's efficient memory management strategy in stage 2.

Result 1 *DM's optimal action in stage 4 moves in the direction of the signal. It is close to the prior when the memory system employed is very imprecise and close to the signal when the memory system employed is very precise.*

Under either system, the posterior belief about the state, and therefore about the optimal action to be taken, is a convex combination of prior μ and signal s_i . Formally, $\hat{a}(s_i) = \frac{p}{p+h_i}\mu + \frac{h_i}{p+h_i}s_i$. A signal below the prior indicates the state is likely to be below μ , hence an optimal action $\hat{a}(s_i) \in (s_i, \mu)$. A signal above the prior indicates the state is likely to be above μ , hence an optimal action $\hat{a}(s_i) \in (\mu, s_i)$. As the precision h_i of memory system i increases, the signal s_i becomes more informative and reliable. DM is then willing to put a higher weight on the signal and a lower weight on the prior.

Result 1 can be used to predict the direction of the choices under either memory system. Because of imperfect retrieval of the experience, those choices will imply some expected losses and an efficient memory management opts for the system yielding higher expected utility. We now characterize the optimal choice of system.

Result 2 *It is optimal to retrieve information with the declarative memory system when the state is extreme and with the procedural memory system when the state is intermediate.*

Formally, there exists a cutoff $x^* \geq 0$ such that $V_P(x) > V_D(x)$ if $x \in (\mu - x^*, \mu + x^*)$ and $V_D(x) \geq V_P(x)$ if $x \notin (\mu - x^*, \mu + x^*)$.³ It comes directly from equation (1) and has an intuitive explanation. Since the information retrieved with the declarative system is precise, it puts a higher weight on the signal (thus a lower weight on the prior) than the procedural system. When the state is close to the prior μ , the utility loss of remembering it imperfectly is, on average, smaller the higher the weight put on the prior in forming the posterior. Hence, the procedural system dominates. Conversely, when the state is far away from μ , it is on average better to put a low weight on the prior and a high weight on the signal, so the declarative system dominates. Stated differently, because the information retrieved is more accurate (but also more costly!) with the declarative system, the loss

³For some parameters (h_D, h_P, c_D, c_P) , we have $V_D(x) > V_P(x)$ for all x .

under that channel is *relatively* smaller if the state departs substantially from the prior belief. Only for extreme states it is then worth spending the extra cost of memory retrieval implied by the declarative system. This optimal memory management policy is illustrated in Figure 2.⁴

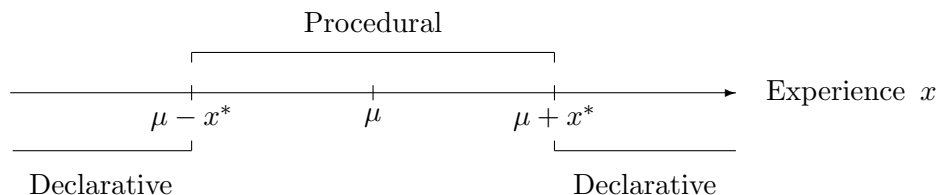


Figure 2. Optimal memory management

4 Discussion

Implications of the theory

The theory outlined above offers testable predictions that can inspire new experimental studies. For our thought experiment, DM will resort to a memory system and make a decision. The experimenter can (i) measure brain activity to identify the memory system that is invoked and (ii) record DM’s behavior. This provides two measures that can be correlated. Results 1 and 2 inform us about the activation pattern and behavior of DM *if* decision-making is efficient. This provides us with a theoretical correlation. We can then compare the theoretical predictions and the observed patterns to determine whether they are in line with each other. This comparison should help us assess whether optimal decision-making is performed in the brain and understand the underlying logic of choices. We open that discussion by reviewing and explaining some implications of our analysis.⁵

We consider those implications from the perspective of an experimenter who is able to control, or at least observe, the experience x (intermediate, extreme, etc.). He can manipulate the distribution from which x is drawn, in particular, the precision p . For example, subjects can sample a fruit (the experience) and be asked to remember the taste

⁴Needless to say, we do not pretend that the brain literally undertakes these sophisticated trade-off calculations (although there is evidence in other contexts that neurons perform approximately Bayesian inferences (see e.g. Barlow, 2001) and compute approximately the expected value of different options (see e.g. Platt and Glimcher, 1999)). In that respect, we take the “as if” stand, typical in economics analyses, to justify the optimization approach. The idea is that what we observe (e.g., behavior and neural correlates) is compatible with the solution of an optimization. Hence, it is valid to say that DM acts as if he was optimizing his expected utility.

⁵Proofs of the implications that do not follow directly from Results 1 and 2 can be found in the Supplementary Information (Text S1).

or sugar content. This fruit may be chosen from a set of local and familiar fruits (high p) or exotic and rare fruits (low p). The experimenter can also affect β by varying the rewards associated to final choices. The costs of memory management c_i can be increased using procedures that add an extra load on working memory (including distractors or peripheral tasks, inducing stress, etc.). Conversely, costs can be decreased with the introduction of cues that facilitate the recollection of events. Also, the experimenter may be able to assess some intrinsic characteristics of DM and obtain some initial estimates of c_i and h_i . For instance, prior testing of experimental subjects can be performed, or the subject pool could be filtered according to specific criteria (diseases, age, etc.). Last, it is possible to prevent substitution between systems by recruiting subjects with brain lesions. The next implications describe what should be observed by the experimenter if our model is a good representation of the memory processes.

Implication 1 *Efficient memory management requires striking events to be retrieved with the declarative system and non striking events to be retrieved with the procedural system.*

As reviewed previously, the neuroscience evidence suggests that the declarative system is invoked to remember striking, episodic events whereas the procedural system is used to remember general aspects and recurrent patterns of tasks. A main result of the paper is to show that these may not be exogenous properties of the memory systems. Instead, they can be the result of an *optimal memory management policy*. So, for example, we remember the details of a clever proof, the ingredients of an awesome desert, and the moment where the music conductor made a mistake. By contrast, we have only a vague idea of what was in a moderately interesting paper, in an uninspiring meal and in an average concert. In terms of our previous example, the theory implies that DM will remember with accuracy an extremely good or bad experience with a product because it will engage the declarative memory system. However, an average experience will not stand out as it will engage the procedural memory system. At the time of retrieving the information, the exact experience will be confounded with other average experiences. Overall, our analysis suggests that the difference in the precision of these memories is due to the *endogenous selection* of different systems to retrieve average vs. extreme experiences.

Note that the result builds on the premise that the declarative and procedural systems are substitutable, an idea that has received experimental support (see the Introduction). This suggests the existence of a mechanism allowing memory systems to be selected depending on the information to be retrieved.

Implication 2 *DM takes decisions closer to the original experience x when he resorts to the declarative system than when he resorts to the procedural system.*

Suppose that DM cannot select which memory system to use due to a lesion or simply because one system is currently overloaded with a different task. A simple comparative

statics between the optimal decisions under each system can be performed to assess behavioral differences. More precisely, consider a case where the experimenter controls x and observes the decision of a lesion patient. When the declarative system has to be used, DM’s behavior tends to hinge closer to the signal than when the procedural system has to be used. In both cases, experiences $x < \mu$ trigger actions that are, on average, below μ and experiences $x > \mu$ trigger actions that are, on average, above μ . However, when comparing the behavior of subjects with different impairments, we notice a systematic bias in decision-making. Formally, the decision taken on average is:

$$E[\hat{a}(s_i)] = \frac{p}{p + h_i} \mu + \frac{h_i}{p + h_i} x$$

and the bias of the expected decision with respect to the original experience is:

$$E[\hat{a}(s_i)] - x = \frac{p}{p + h_i} (\mu - x)$$

As h_i , the precision of the memory increases, the average decision comes closer to the original experience (formally, $|E[\hat{a}(s_i)] - x|$ is decreasing in h_i). Therefore, subjects exhibiting an impaired procedural memory and resorting exclusively to the declarative memory (the high precision system) are more likely to behave in accordance with the original experience. By contrast, subjects with an impaired declarative memory will tend to depart more from the original experience. This implication indicates that subjects with deficits of the hippocampus and related structures should perform poorly at remembering “whether they liked the product”. This prediction is in line with experimental evidence (see Eichenbaum et al (1990)). By contrast, subjects with a deficit of the striatum should recall all types of experiences equally well.⁶

Implication 3 *DM employs more often the procedural system when the cost of the declarative system or the precision of the procedural system increase, and when the cost of the procedural system or the precision of the declarative system decrease. Also, when the cost of one system increases, it becomes relatively more beneficial to resort to the other system if the cost of that other system is already low.*

Formally, x^* increases in c_D , decreases in c_P , and for precision values large enough, it decreases in h_D and increases in h_P (a sufficient condition is $h_P > p$). The comparative statics with respect to costs are natural: a system is relatively less likely to be employed the higher its associated cost. As for h_i , one should remember that higher precision of a memory system is, in general, good: it implies a better recollection of the experience and

⁶Note that our task consists in retrieving information about a fact. Many patient studies involve tasks consisting in retrieving information about a skill. They correspond to a different model with a different loss function.

therefore a better sense of which action to take. It is again natural that if one system becomes relatively more accurate then, at the margin, it is more likely to be invoked. Finally, following a cost increase of one system, it is relatively more desirable to switch to the other system when that other system is already efficient. (formally, $\partial x^*/\partial c_D \partial c_P < 0$). These predictions are, in principle, testable. Subjects with different quality of memory should exhibit different behaviors and different activation patterns. For instance, the capacity of working memory is known to increase over time in children (see Gathercole et al. (2004)) and to decrease over time for elderly subjects (see Salthouse (1994)). A higher working memory capacity corresponds to a lower cost of the declarative memory. Combining these differences across ages with our theory, children should resort more to the declarative system as they grow (i.e., use the declarative system to remember a larger range of experiences) whereas adults should resort more to the procedural system as they get old (i.e., use the declarative system only for very striking experiences).

Implication 4 *When suboptimal actions are more costly, DM employs less often the procedural system and he is less likely to substitute memory systems as their cost change.*

The parameter β reflects the sensitivity to losses, with a larger β implying a steeper loss function. When β increases, it becomes more important to recall information accurately. Therefore, DM should use the declarative system more often (formally, $\partial x^*/\partial \beta < 0$). Thus, a testable implication is that subjects should, at the margin, resort relatively more often to the declarative system when the experimenter increases the reward for an accurate recall. The result has also implications for the design of institutions. If incentives shape the way DM retains and retrieves information, some inefficiencies attributed to phenomena under the umbrella of “bounded rationality” may be better understood and corrected.

Moreover, from Implication 3, we know that as the cost of one memory system increases, the likelihood of a substitution with the other system also increases. Interestingly, this substitution effect is as small as the cost of suboptimal choices increases. That is, subjects who lose more from imperfect recall are also less sensitive to variations in the cost of a system: they tend to switch less often when the system becomes more effortful. Manipulating the cost of working memory may help testing this prediction.

Implication 5 *When the state is drawn from a distribution with large variance ($p \rightarrow 0$), either the procedural system is always used or the declarative system is always used.*

When the precision of the true state is arbitrarily small, DM relies exclusively on the signal, that is, the decision is always s_i . The signal is closer to the experience under the declarative system but also more costly. Thus, the system offering the best trade-off between cost and precision will *always* be invoked.

This last prediction is also testable. In experiments where DM has a priori a flat belief (e.g., in new or unpredictable situations), some subjects will resort to the procedural

system while others will invoke the declarative system. However, the recording pattern for a given subject will not change across experiments (product A vs. product B) or as a function of the realized state (I liked it a lot vs. I liked it moderately). So, for instance, consider a child who learns how to read and has no prior knowledge of the alphabet. When exposed to a letter for the first time, he may forget it, making mistakes repeatedly. While this phenomenon is often linked to attention, it may simply be the case that the child is (optimally!) using the low-cost procedural memory and learns more slowly than children who use the (effortful) declarative memory. Even though our model is not designed to make predictions about learning (which would require a dynamic framework), it points to the existence of a causal relationship between the ability to learn/memorize and the intrinsic features of the two memory systems.

Final comments

The objective of this study was to build a theory able to unveil *causal* relationships between memory systems and the characteristics of the information retrieved. The theory argues that such causal relationships emerge as the result of an *optimal memory management* policy. The type of information memorized by each system emerges as the optimal solution of a trade-off between costs and precision of memories. The logic of the argument is summarized in Figure 3. The theory also offers predictions that can be tested as reviewed in Implications 1 to 5. We believe this contribution is key for experimental practices. Indeed, theoretical models are flexible. They can help formulate testable paradigms and refine experimental designs.

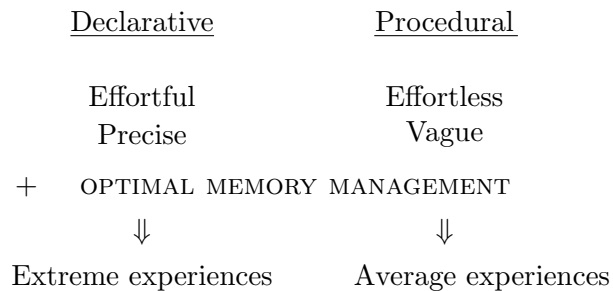


Figure 3. Summary

This study also provides an example of how the methodology in microeconomic theory can help understand biological phenomena and offer testable predictions. Yet, it is incomplete as it abstracts from important considerations such as other features of the declarative and procedural memory systems as well as the interactions of these systems with other systems in the brain. An obvious area for improvement would be to enrich the current model with these considerations.

Of special interest is the recent evidence on competition and cooperation between memory systems. For instance, Gold (2004) shows that competition can be counter-productive: performance is increased in a task if the ‘efficient’ system for that task is boosted, and/or the ‘inefficient’ system is inhibited. The idea that systems can cooperate has also received support (Willingham (1998), Gold (2004) and Colombo and Gold (2004)). It would be also interesting to incorporate these effects in our formal model in order to understand better the “externalities” that different memory systems impose on each other.

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Appendix

Proof of Implication 3. At equilibrium $V_P(\mu - x^*) = V_D(\mu - x^*)$. Differentiating this expression with respect to c_P we have,

$$-\left(\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*)\right) \frac{\partial x^*}{\partial c_P} = -\frac{\partial V_P}{\partial c_P}(\mu - x^*).$$

We also have $\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*) \propto h_D - h_P > 0$ and $\frac{\partial V_P}{\partial c_P}(\mu - x^*) = -1$, therefore, $\frac{\partial x^*}{\partial c_P} < 0$. As c_P increases, $\mu - x^*$ increases and $\mu + x^*$ decreases: the region in which DM uses the procedural memory shrinks. Similarly,

$$-\left(\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*)\right) \frac{\partial x^*}{\partial c_D} = \frac{\partial V_D}{\partial c_D}(\mu - x^*)$$

and given $\frac{\partial V_D}{\partial c_D}(\mu - x^*) = 1$, we have $\frac{\partial x^*}{\partial c_D} > 0$. With respect to precisions, we have

$$\begin{aligned} -\left(\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*)\right) \frac{\partial x^*}{\partial h_P} &= -\frac{\partial V_P}{\partial h_P}(\mu - x^*), \\ -\left(\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*)\right) \frac{\partial x^*}{\partial h_D} &= \frac{\partial V_D}{\partial h_D}(\mu - x^*) \end{aligned}$$

Note that $\frac{\partial V_i}{\partial h_i}(\mu - x^*) \propto h_i - p + p^2 x^* > 0$ if $h_i > p$ in which case $\frac{\partial x^*}{\partial h_P} > 0$ and $\frac{\partial x^*}{\partial h_D} < 0$. Differentiating the first equation above with respect to c_D , we also have:

$$\left(\frac{\partial V_P^2}{\partial x^2}(\mu - x^*) - \frac{\partial^2 V_D}{\partial x^2}(\mu - x^*)\right) \frac{\partial x^*}{\partial c_D} \frac{\partial x^*}{\partial c_P} = \left(\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*)\right) \frac{\partial^2 x^*}{\partial c_P \partial c_D}$$

where $\frac{\partial V_P^2}{\partial x^2}(\mu - x^*) - \frac{\partial^2 V_D}{\partial x^2}(\mu - x^*) \propto h_D - h_P > 0$ yielding $\frac{\partial^2 x^*}{\partial c_P \partial c_D} < 0$.

Proof of Implication 4. Differentiating the equilibrium condition $V_P(\mu - x^*) = V_D(\mu - x^*)$ with respect to β we have,

$$-\left(\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*)\right) \frac{\partial x^*}{\partial \beta} = -\left(\frac{\partial V_P}{\partial \beta}(\mu - x^*) - \frac{\partial V_D}{\partial \beta}(\mu - x^*)\right)$$

At equilibrium, we have $\frac{\partial V_P}{\partial \beta}(\mu - x^*) - \frac{\partial V_D}{\partial \beta}(\mu - x^*) = (c_P - c_D)/\beta < 0$ and therefore $\frac{\partial x^*}{\partial \beta} < 0$. Differentiating a second time with respect to c_D :

$$\left(\frac{\partial^2 V_P}{\partial x^2}(\mu - x^*) - \frac{\partial^2 V_D}{\partial x^2}(\mu - x^*)\right) \frac{\partial x^*}{\partial \beta} \frac{\partial x^*}{\partial c_D} - \frac{1}{\beta} = \left(\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*)\right) \frac{\partial^2 x^*}{\partial \beta \partial c_D}$$

yielding $\frac{\partial^2 x^*}{\partial \beta \partial c_D} < 0$. Differentiating now a second time with respect to c_P :

$$\left(\frac{\partial^2 V_P}{\partial x^2}(\mu - x^*) - \frac{\partial^2 V_D}{\partial x^2}(\mu - x^*) \right) \frac{\partial x^*}{\partial \beta} \frac{\partial x^*}{\partial c_P} + \frac{1}{\beta} = \left(\frac{\partial V_P}{\partial x}(\mu - x^*) - \frac{\partial V_D}{\partial x}(\mu - x^*) \right) \frac{\partial^2 x^*}{\partial \beta \partial c_P}$$

yielding $\frac{\partial^2 x^*}{\partial \beta \partial c_P} > 0$.

Proof of Implication 5. When $p \rightarrow 0$, we have $V_i(x) = -\frac{\beta}{h_i} - c_i$ and $V_D(x) - V_P(x) = -\frac{\beta}{h_D h_P}(h_P - h_D) + c_P - c_D$, which is independent of x .